UK Maths Trust

Intermediate Mathematical Challenge

Organised by the United Kingdom Mathematics Trust

proudly sponsored by XTX MARKETS

Solutions and investigations

29 January 2025

These solutions augment the shorter solutions also available online. The shorter solutions in many cases omit details. The solutions given here are full solutions, as explained below. In some cases alternative solutions are given. There are also many additional problems for further investigation. We welcome comments on these solutions and the additional problems. Please send them to challenges@ukmt.org.uk.

The Intermediate Mathematical Challenge (IMC) is a multiple-choice paper. For each question, you are presented with five options, of which just one is correct. It follows that sometimes you can find the correct answers by working backwards from the given alternatives, or by showing that four of them are not correct. This can be a sensible thing to do in the context of the IMC.

However, this does not provide a full mathematical explanation that would be acceptable if you were just given the question without any alternative answers. So for each question we have included a complete solution which does not use the fact that one of the given alternatives is correct. Thus we have aimed to give full solutions with each step explained (or, occasionally, left as an exercise). We therefore hope that these solutions can be used as a model for the type of written solution that is expected when a complete solution to a mathematical problem is required (for example, in the Intermediate Mathematical Olympiad and similar competitions).

Enquiries about the Intermediate Mathematical Challenge should be sent to:

IMC, challenges@ukmt.org.uk   *www.ukmt.org.uk*

supported by Overleaf

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
A	D	E	D	E	C	B	A	A	C	A	E	E	A	D	B	D	E	D	B	B	D	D	C	B

1. Which one of the following expressions has a value closest to 0?

A $2 \times 5 - 8 \times 3$ B $3 \times 4 - 7 \times 4$ C $4 \times 3 - 6 \times 5$ D $5 \times 2 - 5 \times 6$ E $6 \times 1 - 4 \times 7$

Solution **A**

We have

$$\begin{aligned} 2 \times 5 - 8 \times 3 &= 10 - 24 = -14, \\ 3 \times 4 - 7 \times 4 &= 12 - 28 = -16, \\ 4 \times 3 - 6 \times 5 &= 12 - 30 = -18, \\ 5 \times 2 - 5 \times 6 &= 10 - 30 = -20, \\ \text{and} \quad 6 \times 1 - 4 \times 7 &= 6 - 28 = -22. \end{aligned}$$

From these calculations we see that $2 \times 5 - 8 \times 3$ is the option that is closest to 0.

For investigation

1.1 The expressions given as the options in this question are the first five terms of the sequence whose nth term is given given by the formula $(n + 1) \times (6 - n) - (9 - n) \times (n + 2)$.

(a) For which value of n is the nth term of this sequence equal to -1000?

(b) Prove that this is a decreasing sequence, all of whose terms are negative.

2. What is the remainder when 2 652 134 is divided by 13?

A 1 B 2 C 3 D 4 E 5

Solution **D**

This question could be answered by doing a long division sum in the usual way. However, when you look at the number 2 652 134, you should notice that 26, 52 and 13 are all multiples of 13, thus leaving a remainder of 4 when the whole number is divided by 4. We therefore set out the calculation as shown below.

We have

$$\begin{aligned} 2\,652\,134 &= 2\,600\,000 + 52\,000 + 130 + 4 \\ &= 13 \times 200\,000 + 13 \times 4000 + 13 \times 10 + 4 \\ &= 13 \times (200\,000 + 4000 + 10) + 4. \end{aligned}$$

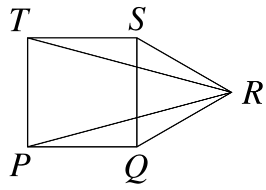
Therefore the remainder when 2 652 134 is divided by 13 is 4.

For investigation

2.1 What is the remainder when 426 349 845 is divided by 7?

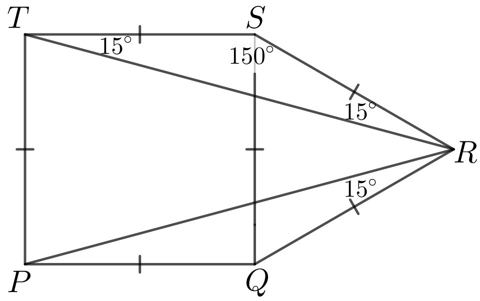
3. The diagram shows a square $PQST$ and an equilateral triangle QRS.
What is the size, in degrees, of angle PRT?

A 10 B 15 C 20 D 25 E 30



Solution **E**

Because $PQST$ is a square, $\angle TSQ = 90°$. Because QRS is an equilateral triangle, $\angle QSR = 60°$. It follows that $\angle TSR = 90° + 60° = 150°$.



The sum of the angles in a triangle is 180°. Therefore, by considering the angles in the triangle TSR, we deduce that

$$\angle STR + \angle SRT = 180° - 150° = 30°. \quad (1)$$

Also, because $PQST$ is a square, $QS = ST$, and because QRS is an equilateral triangle $QS = SR$. It follows that $ST = SR$. Therefore the triangle STR is isosceles. Hence

$$\angle SRT = \angle STR. \qquad (2)$$

From (1) and (2) it follows that $\angle SRT = \frac{1}{2}(30°) = 15°$. Similarly $\angle PRQ = 15°$.

It follows that

$$\angle PRT = \angle QRS - (\angle PRQ + \angle SRT) = 60° - (15° + 15°) = 30°.$$

For investigation

3.1 Find the length of TR in terms of the side length of the square and the equilateral triangle.

4. All Felix's cats are normal cats. Together, they have 12 more legs than they have tails. In total, how many ears do they have?

A 2 B 4 C 6 D 8 E 10

Solution **D**

Each normal cat has 3 more legs than it has tails. Because there are 12 more legs than tails, there are $12 \div 3 = 4$ cats. Between them, these cats have $4 \times 2 = 8$ ears.

For investigation

4.1 Among a group of normal cats there are n more legs than tails. Find a formula in terms of n for the total number of ears these cats have.

5. What is $5 \div (((5 \div 5) \div (5 \div 5)) \div 5)$?

A $\frac{1}{25}$ B $\frac{1}{5}$ C 1 D 5 E 25

SOLUTION **E**

We have

$$(5 \div 5) \div (5 \div 5) = 1 \div 1 = 1.$$

Therefore

$$((5 \div 5) \div (5 \div 5)) \div 5 = 1 \div 5 = \frac{1}{5}.$$

Hence

$$5 \div (((5 \div 5) \div (5 \div 5)) \div 5) = 5 \div \frac{1}{5} = 5 \times \frac{5}{1} = 25.$$

FOR INVESTIGATION

5.1 Find the values of the following expressions.

(a) $7 \div (((7 \div 7) \div (7 \div 7)) \div 7)$,

(b) $(5 \div (5 \div 5)) \div ((5 \div 5) \div 5)$.

6. Owen chooses a positive integer n so that $3n + 7$ is an even integer.
Which of these is an odd integer?

A $n - 1$ B $n + 3$ C $3n + 2$ D $4n$ E $5n + 3$

SOLUTION **C**

METHOD 1

We can immediately exclude option D because $4n$ is even whether n is odd or even.

Because $3n + 7$ is even, the number $3n$ is odd. Therefore n is odd. It follows that $n - 1$ and $n + 3$ are even. Also $5n$ is odd and hence $5n + 3$ is even. However $3n$ is odd and therefore $3n + 2$ is odd. It follows that the correct option is C.

METHOD 2

We note that $3n + 2 = (3n + 7) - 5$. Therefore, as $3n + 7$ is even, it follows that $3n + 2$ is odd. In the context of the IMC we can assume that just one of the options is correct, and hence that the correct option is C.

FOR INVESTIGATION

6.1 How many even integers are there in the set $\{5n+7,\ n^2+2,\ n^3-1,\ n^2+n^3,\ n+n^2+n^3+n^4\}$, (a) when n is an odd integer, and (b) when n is an even integer?

7. "I can swim faster than you," said the dolphin to the shark.
"That is not true," said the shark to the dolphin.
"You are both wrong," said the octopus to the dolphin and the shark.
"You are right," said the starfish to the octopus.
How many of the dolphin, shark, octopus and starfish were telling the truth?

A 0 B 1 C 2 D 3 E 4

SOLUTION **B**

If the dolphin were telling the truth, then what the shark said was false. Similarly, if what the shark said were true, then the dolphin was not telling the truth.

It follows that exactly one of the dolphin and the shark was telling the truth.

Therefore, what the octopus said was false. It follows that also what the starfish said was false.

Therefore we can deduce that exactly one of them was telling the truth.

Note that the information we are given in this question enables us to work out that the single truth teller was either the dolphin or the shark, but does not enable us to deduce which of these two it was.

FOR INVESTIGATION

7.1 "I can swim faster than you," said the hake to the haddock.
"I can swim faster than you," said the hoki to the haddock.
"You are both wrong," said the herring to the hake and the hoki.
"You are right," said the haddock to the herring.
"Just one of you is right," said the halibut to the hake and the hoki.

How many of them were right?

8. What is the value of $34\frac{1}{7} \div 17\frac{1}{14}$?

A 2 B $2\frac{1}{17}$ C $2\frac{1}{14}$ D $2\frac{1}{7}$ E $2\frac{1}{2}$

SOLUTION **A**

We note that $34 = 2 \times 17$ and $\frac{1}{7} = 2 \times \frac{1}{14}$. Therefore

$$34\tfrac{1}{7} = 34 + \tfrac{1}{7} = 2 \times 17 + 2 \times \tfrac{1}{14} = 2 \times \left(17 + \tfrac{1}{14}\right) = 2 \times 17\tfrac{1}{14}.$$

It follows that $$34\tfrac{1}{7} \div 17\tfrac{1}{14} = 2.$$

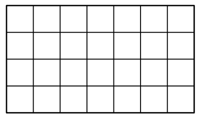
FOR INVESTIGATION

8.1 What are the values of

(a) $87\frac{1}{8} \div 29\frac{1}{24}$.

(b) $69\frac{3}{5} \div 17\frac{2}{5}$?

9. Lottie wants to colour two rows and two columns of the 4×7 grid shown so that the coloured rows do not touch and also the coloured columns do not touch.



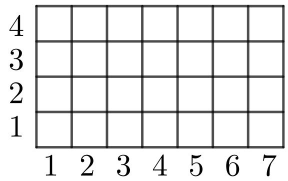
In how many ways can she do this?

A 45 B 42 C 40 D 35 E 30

Solution **A**

Lottie can choose the two rows to colour independently from her choice of the two columns. Therefore the number of ways she can choose the rows and columns is the number of ways she can chose the two rows multiplied by the number of ways she can choose the two columns.

In the diagram on the right we have numbered the rows and the columns so that we can more easily refer to them.



We count the pairs of non-touching rows according to which is the lower of the two rows.

If row 1 is the lower row, the second row could be either row 3 or row 4. If row 2 is the lower row, the second row has to be row 4. It is not possible for either row 3 or row 4 to be the lower row. This gives a total of 3 choices, rows 1 and 3, or rows 1 and 4, or rows 2 and 4.

We count the pairs of non-touching columns according to which column is on the left of the pair.

If column 1 is on the left, the second column could be any of columns 3, 4, 5, 6 and 7, giving 5 choices. If column 2 is on the left, the second column could be any of columns 4, 5, 6 and 7, giving 4 choices. Continuing in this way, we find that there are $5 + 4 + 3 + 2 + 1 = 15$ ways to choose two non-touching columns.

It follows that the total number of ways Lottie can choose the rows and columns is $3 \times 15 = 45$.

For investigation

9.1 Another method for counting the number of pairs non-touching rows would be to count the total number of pairs of rows and then subtract the number of touching pairs. The same method could be used to count the pairs of non-touching columns.

Check that this second method gives the same answer as in the solution above.

9.2 Suppose that Lottie wants to colour two non-touching rows, and two non-touching columns, of an $m \times n$ grid, where m and n are integers greater than 2.

(a) Find a formula, in terms of m and n for the number of ways in which Lottie can do this.

(b) Check that your formula has the value 45 when $m = 4$ and $n = 7$.

9.3 In how many ways could Lottie choose three columns from the 4×7 grid so that no two of the columns are touching?

10. What is the value of $\sqrt{2025^2 - 2024 - 2025}$?

A 2022 B 2023 C 2024 D 2025 E 2026

SOLUTION **C**

In the context of the IMC, where you are not permitted to use a calculator, you could not be expected to answer this question by squaring 2025, subtracting 2024 and 2025, and then finding the square root of the resulting seven-digit number.

So there should be a better method. The actual numbers in this question are something of a distraction. So we begin by making a substitution which changes the question from arithmetic to algebra.

We put $n = 2025$. Then

$$\begin{aligned} 2025^2 - 2004 - 2005 &= n^2 - (n-1) - n \\ &= n^2 - 2n + 1 \\ &= (n-1)^2. \end{aligned}$$

It follows that

$$\begin{aligned} \sqrt{2025^2 - 2024 - 2025} &= n - 1 \\ &= 2025 - 1 \\ &= 2024. \end{aligned}$$

FOR INVESTIGATION

10.1 Find the value of

$$\sqrt{2025^2 + 2025 + 2026}$$

without using a calculator.

10.2 Find the value of

$$\sqrt[3]{999^3 + 3000 \times 999 + 1}$$

without using a calculator.

10.3 For which values of n is it true that

$$\sqrt{(n-1)^2} = n - 1?$$

11. Emily multiplied two integers together. Her answer was 360. Finley increased each of Emily's integers by one and then multiplied the two new integers together. His answer was 400.
What was the sum of Emily's two integers?

A 39 B 40 C 42 D 45 E 48

SOLUTION **A**

Let Emily's integers be be m and n. When Emily multiplies these she gets the answer 360. Therefore

$$mn = 360. \quad (1)$$

Finley multiplies $m + 1$ and $n + 1$ and obtains 400. Hence

$$(m + 1)(n + 1) = 400.$$

That is

$$mn + m + n + 1 = 400. \quad (2)$$

From (1) and (2)

$$360 + m + n + 1 = 400$$

and therefore

$$m + n = 400 - 360 - 1 = 39.$$

Therefore, the sum of Emily's two integers is 39.

FOR INVESTIGATION

11.1 In the above solution we have found the value of $m + n$ without finding the values of m and n. Find their values.

11.2 Emily chose three positive integers and multiplied them together. Her answer was 360. Finley increased each of Emily's integer by one and then multiplied the three new integers together. His answer was 550.

Find Emily's three integers.

12. Jane's farm is home to many cats and dogs. At the start of the week the ratio of cats to dogs was 3 : 5. Then 32 cats, but no dogs, arrived and the ratio became 5 : 3.
How many dogs are there on Jane's farm?

A 12 B 18 C 24 D 25 E 30

SOLUTION **E**

At the beginning of the week the ratio of cats to dogs was 3 : 5. We may therefore suppose that at the beginning of the week there were $3n$ cats and $5n$ dogs on Jane's farm, where n is some positive integer.

It follows that at the end of the week there were $3n + 32$ cats and $5n$ dogs on the farm. They were then in the ratio 5 : 3. Therefore

$$\frac{3n + 32}{5n} = \frac{5}{3}.$$

It follows that

$$9n + 96 = 25n.$$

Hence

$$96 = 16n.$$

Therefore, $n = 6$. Hence there were $5 \times 6 = 30$ dogs on Jane's farm.

FOR INVESTIGATION

12.1 In the following week some dogs but no cats arrived. At the end of the week the ratio of cats to dogs was 2 : 3. How many dogs arrived during that week?

13. The product of the first three numbers in the box below is equal to the product of the last three numbers. What is the value of x?

x	120	496	360	48

A 72 B 84 C 96 D 128 E 144

SOLUTION **E**

We have $x \times 120 \times 496 = 496 \times 360 \times 48$. Therefore

$$x = \frac{496 \times 360 \times 48}{120 \times 496} = \frac{360 \times 48}{120} = 3 \times 48 = 144.$$

FOR INVESTIGATION

13.1 The product of the first three numbers in the box below is equal to the product of the last three numbers. Both $x \neq 0$ and $y \neq 0$. What is the ratio $x : y$?

x	120	496	360	y

14. What is the sum of the recurring decimals $0.\dot{1} + 0.\dot{2} + 0.\dot{3} + 0.\dot{4}$?

A $1.\dot{1}$ B $1.\dot{1}\dot{0}$ C 1.1 D $1.\dot{0}\dot{1}$ E 1

SOLUTION **A**

The recurring decimals $0.\dot{1}$, $0.\dot{2}$, $0.\dot{3}$ and $0.\dot{4}$ are equal to $\frac{1}{9}$, $\frac{2}{9}$, $\frac{3}{9}$ and $\frac{4}{9}$, respectively. [You are asked to prove this in Problem 14.1.] Hence,

$$0.\dot{1} + 0.\dot{2} + 0.\dot{3} + 0.\dot{4} = \frac{1}{9} + \frac{2}{9} + \frac{3}{9} + \frac{4}{9} = \frac{10}{9} = 1\tfrac{1}{9} = 1.\dot{1}.$$

FOR INVESTIGATION

14.1 Show that $0.\dot{1}$, $0.\dot{2}$, $0.\dot{3}$ and $0.\dot{4}$ are equal to $\frac{1}{9}$, $\frac{2}{9}$, $\frac{3}{9}$ and $\frac{4}{9}$, respectively.

14.2 What is the sum of the recurring decimals $0.\dot{5} + 0.\dot{6} + 0.\dot{7} + 0.\dot{8} + 0.\dot{9}$?

14.3 Express the recurring decimal $0.\dot{9}$ in its simplest form.

14.4 Write the recurring decimals $1.\dot{1}\dot{0}$ and $1.\dot{0}\dot{1}$ as fractions in their lowest terms.

15. What is the value of $\frac{3^6 - 3^4}{2^9 - 2^3}$?

A $\frac{9}{32}$ B $\frac{9}{16}$ C $\frac{9}{12}$ D $\frac{9}{7}$ E $\frac{9}{5}$

SOLUTION **D**

$$\frac{3^6 - 3^4}{2^9 - 2^3} = \frac{3^4(3^2 - 1)}{2^3(2^6 - 1)} = \frac{81 \times (9 - 1)}{8 \times (64 - 1)} = \frac{81 \times 8}{8 \times 63} = \frac{81}{63} = \frac{9 \times 9}{7 \times 9} = \frac{9}{7}.$$

FOR INVESTIGATION

15.1 What is the value of

$$\frac{5^5 - 5^3}{3^5 - 3}?$$

15.2 Find the all the positive integers n for which

$$\frac{7^n - 7^{n-2}}{2^{n+2} - 2^n} = 49.$$

16. David likes books. He only ever pays 99p or £1.99 for an electronic book. In the last few months he has spent £56.56 on electronic books. How many electronic books costing £1.99 has he bought in that time?

A 8 B 13 C 15 D 17 E 23

Solution **B**

Suppose that David bought x electronic books costing 99p each and y electronic books costing £1.99 each.

Each book cost him either 1p less than £1 or 1p less than £2.

David has spent only £56.56. Even if all the books he bought cost 99p, he can't have bought 144 or more. So the reason why he spent 44 pennies less than a whole number of pounds can only have been that he bought 44 books.

It follows that

$$x + y = 44. \qquad (1)$$

These books cost him £57 less 44p. Therefore

$$x + 2y = 57. \qquad (2)$$

By subtracting equation (1) from equation (2), we find that

$$y = 13.$$

Therefore David bought 13 books costing £1.99.

For investigation

16.1 Karen likes electronic books even more than David. She also only ever pays 99p or £1.99 for an electronic book.

In the last few months she spent £100 buying electronic books.

How many electronic books did she buy during this period?

16.2 Danny buys scratch cards to support a charity. He pays 99p for each card.

If he gets a winning card, he is given a prize of £1 and he also gets back the 99p he paid for the card.

Yesterday he bought some cards and had a net loss of £8.89.

What is the smallest number of cards he could have bought?

17. Dark green paint is made by mixing blue and yellow paint such that 60% is blue.
Light green paint is made by mixing blue and yellow paint such that 60% is yellow.
Pablo mixes dark and light green paint such that 60% of the mixture is dark paint.
What is the ratio of blue to yellow in Pablo's paint?

A 2 : 3 B 12 : 13 C 1 : 1 D 13 : 12 E 3 : 2

SOLUTION **D**

The light green paint is made up of 60% of yellow paint and hence 40% of blue paint.

Therefore, because Pablo's mixture is made up of 60% dark green paint and hence 40% light green paint, the proportion of blue paint in the mixture is

$$\begin{aligned}\left(\frac{60}{100} \times 60 + \frac{40}{100} \times 40\right)\% &= (36 + 16)\% \\ &= 52\%.\end{aligned}$$

Hence the mixture contains 48% yellow paint.

It follows that the ratio of blue paint to yellow paint is 52 : 48. This is equal to 13 : 12.

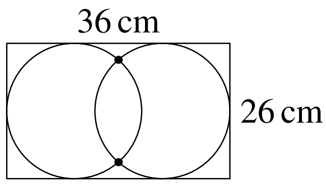
FOR INVESTIGATION

17.1 If Pablo had mixed dark and light green paint in the ratio 2 : 1, what would have been the percentage of blue paint in the mixture?

17.2 If Pablo wanted to end up with a mixture in which the ratio of blue paint to yellow paint was 9 : 11, what percentage of dark green paint should he put in?

17.3 If Pablo wanted to end up with a mixture in which the ratio of blue paint to yellow paint was 2 : 1, could he obtain this by mixing dark green paint with light green paint?

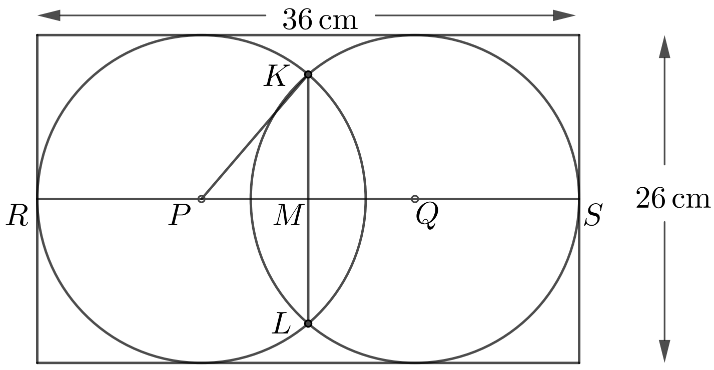
18. Two circles each touch three sides of a rectangle measuring 36 cm × 26 cm, as is shown in the diagram, which is not drawn to scale.
What is the distance between the two points where the circles intersect?



A 18 cm B 20 cm C 21 cm D 22 cm E 24 cm

Solution **E**

Let P and Q be the centres of the circles, and let K and L be the points where the circles intersect, as shown in the diagram.



By the symmetry of the figure, the line that goes through P and Q is parallel to the horizontal edges of the rectangle. Let R and S be the points where this line meets the vertical edges of the rectangle, as shown.

The circles touch the vertical edges of the rectangle at the points R and S. [You are asked to prove this in Problem 18.2.]

Also, the line joining K and L is perpendicular to PQ. We let M be the point where these two lines meet.

Because the height of the rectangle is 26 cm, each circle has radius 13 cm. Hence

$$PQ = RS - RP - QS = 36\,\text{cm} - 13\,\text{cm} - 13\,\text{cm} = 10\,\text{cm}.$$

By the symmetry of the figure $PM = MQ$. Hence PM has length 5 cm.

PK is a radius of the circle with centre P. Hence PK has length 13 cm.

Let the length of KM be x cm.

Then, by Pythagoras' Theorem, applied to the right-angled triangle PMK, $5^2 + x^2 = 13^2$. Hence $x^2 = 13^2 - 5^2 = 169 - 25 = 144$. Therefore $x = 12$. That is, KM has length 12 cm.

Again, by the symmetry of the figure $LM = KM$. We therefore conclude that KL has length 24 cm.

For investigation

18.1 Find the length of RK.

18.2 Prove that the line which goes through the centre of the circles meets the vertical edges of the rectangle at the points where the circles touch these edges.

19. Rob, Rog and Roy all painted some fence posts one day. Rob painted 45, Rog painted 51 and Roy painted 48. One of them painted twice as many posts in the morning as he did in the afternoon, a second one painted three times as many in the morning as in the afternoon and the third one painted four times as many in the morning as in the afternoon. Who painted the most fence posts in the morning?

A Rob
B Rog
C Roy
D Rob and Roy, who painted the same number
E Rob, Rog and Roy, who all painted the same number

SOLUTION **D**

Suppose that the person who painted twice as many fence posts in the morning as in the afternoon painted n of them in the afternoon. Then they painted $2n$ fence posts in the morning and hence $2n + n = 3n$ in the whole day. Therefore the number of fence posts they painted was a multiple of 3. Hence this person could have been any of the three, as 45, 51 and 48 are all multiples of 3.

Similarly, the number of fence posts painted by the person who painted three times as many in the morning as in the afternoon must have been a multiple of 4. Hence this person was Roy, as 48 is a multiple of 4, but 45 and 51 are not.

We deduce that of the 48 fence posts Roy painted, he painted 36 in the morning and 12 in the afternoon.

Also, the number of fence posts painted by the person who painted four times as many in the morning as in the afternoon must have been a multiple of 5. Hence this person was Rob.

We deduce that of the 45 fence posts Rob painted, he painted 36 in the morning and 9 in the afternoon.

This leaves Rog as the person who painted twice as many fence posts in the morning as in the afternoon.

Hence of the 51 fence posts that he painted, Rog painted 34 in the morning and 17 in the afternoon.

Since Rob and Roy each painted 36 fence posts in the morning, whereas Rog painted only 34, the correct option is D.

FOR INVESTIGATION

19.1 On another day Rob, Rog and Roy were painting gates.

The numbers of gates painted by Rob, Roy and Rog were in the ratio $4 : 3 : 2$.

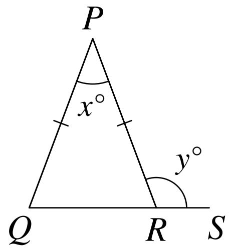
They each painted twice as many gates in the morning as in the afternoon.

What is the smallest total number of gates that they could have painted that day?

20. The triangle PQR is isosceles with $PQ = PR$. The point S lies on the extension of the line QR as shown.
$\angle RPQ = x°$ and $\angle SRP = y°$, where x, y and $\frac{y}{x}$ are integers.

What is the largest possible value of $\frac{y}{x}$?

A 36 B 23 C 17 D 8 E 5



SOLUTION **B**

Because angles on a straight line add up to 180°, we have $\angle PRQ = (180 - y)°$.

Because the triangle PQR is isosceles with $PQ = PR$, we can now deduce that $\angle PQR = \angle PRQ = (180 - y)°$.

Therefore, because the angles in a triangle add up to 180°, it follows that

$$x + (180 - y) + (180 - y) = 180.$$

This last equation may be rearranged to give

$$2y = 180 + x,$$

and hence

$$\frac{y}{x} = \frac{90}{x} + \frac{1}{2}.$$

For $\frac{y}{x}$ to be as large as possible, x must be as small as possible.

For $x = 1, 2$ and 3, $\frac{90}{x}$ is an integer, and hence $\frac{y}{x}$ is not an integer. However, for $x = 4$,

$$\frac{y}{x} = \frac{90}{4} + \frac{1}{2} = \frac{45}{2} + \frac{1}{2} = 23.$$

Hence this is the largest possible value of $\frac{y}{x}$.

FOR INVESTIGATION

20.1 What is the value of y in the case where $\frac{y}{x} = 23$?

20.2 (a) What are the values of x and y for which $\frac{y}{x}$ takes its smallest possible value?

(b) What is the smallest possible value of $\frac{y}{x}$?

21. What is the remainder when 8^8 is divided by 5?

A 0 B 1 C 2 D 3 E 4

SOLUTION **B**

We can work out the remainder when 8^8 is divided by 5 by calculating its units (ones) digit. We use the fact that the numbers $a \times b$ and (the units digit of a) $\times$ (the units digit of b) have the same units digit.

METHOD 1

$8^2 = 64$. Therefore the units digit of 8^2 is 4. It follows that 8^3, which is the same as $8^2 \times 8$ has the same units digit as 4×8. Since $4 \times 8 = 32$, we deduce that the units digit of 8^3 is 2. Similarly, the units digit of 8^4 is the same as the units digit of 8×2, namely 6. And the units digit of 8^5 is the same as that of 8×6 which is 8. We are back to where we began.

From this we see that the units digits of 8^n, for $n = 1, 2, 3, \ldots$ form the periodic sequence $8, 4, 2, 6, 8, 4, 2, 6, 8, \ldots$ that repeats every fourth term.

It follows that the units digit of 8^8 is 6. Therefore the remainder when 8^8 is divided by 5 is the same as the remainder when 6 is divided by 5. So the remainder is 1.

METHOD 2

Because 8 and 3 have the same remainder when divided by 5, so also do 8^8 and 3^8. [This is a special case of a general fact about remainders which you are asked to prove in Problem 21.2.]

Now $3^8 = (3^4)^2 = ((3^2)^2)^2 = (9^2)^2 = 81 \times 81$. From this we see that the units digit of 3^8 is 1. It follows that the remainder when both 3^8 and 8^8 are divided by 5 is 1.

FOR INVESTIGATION

21.1 What is the remainder when 8^{2024} is divided by 5?

21.2 Prove that the units digit of $a \times b$ is the same as the units digit which is obtained when the units digit of a is multiplied by the units digit of b.

21.3 Let a and b be positive integers which leave the same remainder when they are divided by the positive integer n.

(a) Prove that a^2 and b^2 leave the same remainder when they are divided by n.

(b) Prove that a^3 and b^3 leave the same remainder when they are divided by n.

(c) Prove that for each positive integer k, the numbers a^k and b^k leave the same remainder when they are divided by n.

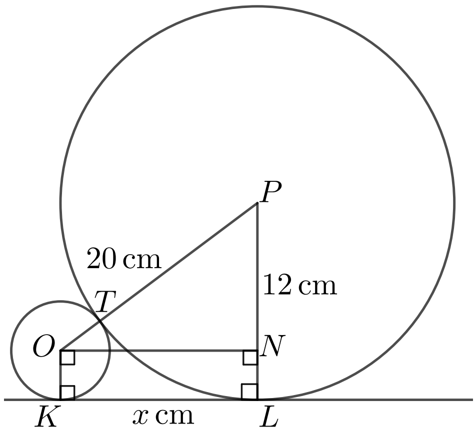
Facts about remainders are covered in *Modular Arithmetic*. You can find out about Modular Arithmetic from the book *Introduction to Number Theory* by C.J.Bradley and a shorter discussion in *A Mathematical Olympiad Primer* by Geoff Smith. Both books are published by UKMT and available from the UKMT online bookshop. You could also do an internet search.

22. A sphere of radius 4 cm and a sphere of radius 16 cm are placed on a horizontal surface so that they touch. What is the distance, in cm, between the points where the spheres touch the surface?

A 12 B 14 C $\sqrt{240}$ D 16 E 20

SOLUTION **D**

The diagram shows a cross section through the centres of the two spheres and perpendicular to the horizontal surface which they touch. The cross sections of the spheres are the circles shown in the diagram.



O is the centre of the sphere with radius 4 cm, P is the centre of the sphere with radius 16 cm, and K and L are the points where the spheres touch the horizontal surface. N is the point on PL such that ON is at right angles to PL. We let x cm be the length of KL.

The line KL is a tangent to both the circles. Hence $\angle OKL = \angle NLK = 90°$. It follows that $KLNO$ is a rectangle.

Hence $ON = KL = x$ cm and $NL = OK = 4$ cm.

The line OP which joins the centres of the circles goes through the point T where they touch. [You are asked to prove this in Problem 22.1.] Hence $OP = OT + TP = 4\,\text{cm} + 16\,\text{cm} = 20\,\text{cm}$.

Also, $PN = PL - NL = PL - OK = 16\,\text{cm} - 4\,\text{cm} = 12\,\text{cm}$.

By applying Pythagoras' Theorem to the right-angled triangle ONP, we have $x^2 + 12^2 = 20^2$. Therefore $x^2 = 20^2 - 12^2 = 400 - 144 = 256 = 16^2$. It follows that $x = 16$. Hence $KL = 16$ cm. This is therefore the distance between the points where the spheres touch the surface.

FOR INVESTIGATION

22.1 Prove that if two circles with centres O and P touch at the point T, then the line OP goes through the point T.

22.2 A sphere of radius a cm and a sphere of radius b cm are placed on a horizontal surface so that they touch.

Find, in terms of a and b the distance between the points where the spheres touch.

22.3 Note that the lengths of the sides of the right-angled triangle PNO, in cm, are 12 cm, 16 cm and 20 cm. So PNO is a scaled up version of the basic 3,4,5 right-angled triangle.

Show that for every right-angled triangle XYZ, it is possible to choose the radii of two spheres so that the triangle PNO in the diagram of this question is congruent to the triangle XYZ.

23. The point P is inside a rectangle. The distance from P to one corner of the rectangle is 5 cm and its distance from the opposite corner is 14 cm. The distance from P to a third corner is 10 cm. What is the distance, in cm, from P to the fourth corner?

A $\sqrt{29}$ B 9 C 10 D 11 E $\sqrt{171}$

SOLUTION **D**

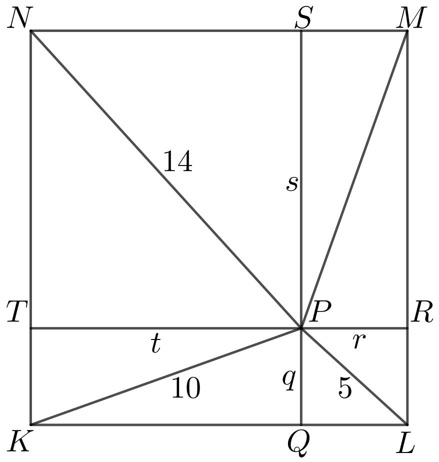
We let K, L, M and N be the vertices of the rectangle.

We let Q, R, S and T be the points where the lines which go through P and are parallel to the edges of the rectangle meet these edges, as shown in the diagram.

We also let the lengths, in cm, of PQ, PR, PS and PT be q, r, s and t, respectively.

The lines QS and RT divide the rectangle into four smaller rectangles.

It follows that $TK = PQ = q$ cm, $QL = PR = r$ cm, $MR = SP = s$ cm and $NS = TP = t$ cm.



PSN is a right-angled triangle with hypotenuse PN of length 14 cm and with shorter sides of lengths s cm and t cm. Therefore, by Pythagoras' Theorem,

$$s^2 + t^2 = 14^2. \qquad (1)$$

Similarly, applying Pythagoras' Theorem to the right-angled triangles TKP and PQL,

$$q^2 + t^2 = 10^2, \qquad (2)$$

and

$$q^2 + r^2 = 5^2. \qquad (3)$$

From equations (1), (2) and (3), we have

$$(q^2 + r^2) + (s^2 + t^2) - (q^2 + t^2) = 5^2 + 14^2 - 10^2.$$

That is,

$$r^2 + s^2 = 25 + 196 - 100 = 121 = 11^2.$$

Hence, by Pythagoras' Theorem applied to the right-angled triangle PRM, we deduce that $PM = 11$ cm. This gives the distance from P to the fourth corner of the rectangle.

FOR INVESTIGATION

23.1 The point Q is inside another rectangle. The distance from Q to one corner of the rectangle is 7 cm and its distance to the opposite corner is 9 cm. The distance of Q to a third corner is 11 cm. What is the distance of Q to the fourth corner?

23.2 Suppose that there is a rectangle and a point inside the rectangle whose distances to one pair of opposite corners are w cm and x cm, and whose distances to the other pair of opposite corners are y cm and z cm. Deduce that $w^2 + x^2 = y^2 + z^2$.

24. Two white roses and a yellow rose costs £5. Two white roses and three red roses cost £10.50. Three yellow roses and two red roses cost £11. What would be the total cost of one red rose, one white rose and one yellow rose?

A £5 B £5.50 C £6 D £6.50 E £7

Solution **C**

Let the cost, in pence, of one white rose, one yellow rose and one red rose be w, y and r, respectively.

From the information given in the question,

$$2w + y = 500, \qquad (1)$$
$$2w + 3r = 1050, \qquad (2)$$

and

$$3y + 2r = 1100. \qquad (3)$$

By adding equations (1), (2) and (3), we obtain

$$5r + 4w + 4y = 2650. \qquad (4)$$

It follows that if we found the value of r, we could deduce the value of $4w + 4y + 4r$ and hence the value of $w + y + r$ which is what we need to answer the question.

By subtracting adding equation (1) from equation (2), we obtain

$$3r - y = 550. \qquad (5)$$

Adding $3 \times$ equation (5) to equation (3) now gives,

$$11r = 2750.$$

It follows that $r = 2750 \div 11 = 250$.

Hence, from equation (4),

$$\begin{aligned} 4r + 4w + 4y &= 5r + 4w + 4y - r \\ &= 2650 - 250 \\ &= 2400. \end{aligned}$$

It follows that $r + w + y = 2400 \div 4 = 600$.

Therefore the cost of one red rose, one white rose and one yellow rose is 600p, that is, £6.

For investigation

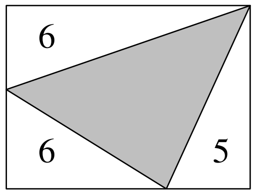
24.1 (a) Find the cost of one yellow rose and the cost of one white rose.

(b) Use your answer to (a) to confirm that the cost of one red rose, one white rose and one yellow rose is £6.

24.2 Two apples and three oranges cost £1.68. Two oranges and three bananas cost £1.19. Two bananas and three apples cost £1.33.

What is the cost of one apple, one orange and one banana?

25. The diagram shows a rectangle which has been divided into four triangles. The areas, in cm^2, of three of the triangles are as shown. What is the area of the shaded triangle?



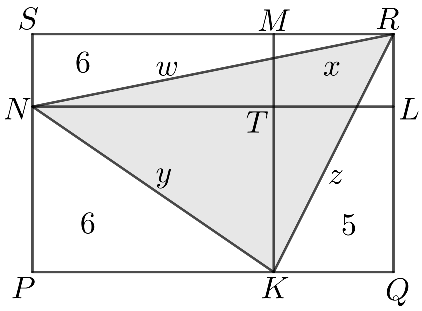
A $11\,\text{cm}^2$ B $13\,\text{cm}^2$ C $14\,\text{cm}^2$ D $15\,\text{cm}^2$ E $16\,\text{cm}^2$

Solution **B**

At first sight it is surprising that it is possible to work out the area of the shaded triangle from the information that is given.

Each of the unshaded triangles forms half of a rectangle. Our first move is to draw in these rectangles and to label the resulting diagram so that we can refer to it, in the hope that this will give us an idea of what to do next.

We let P, Q, R and S be the vertices of the rectangle, and K and N be the vertices of the shaded triangle that are on the edges PQ and SP of $PQRS$, as shown in the diagram.



We let L and M be the points where the lines through N and K parallel to the edges of $PQRS$ meet QR and RS respectively. We also let T be the point where NL meets KM.

We let w, x, y and z be the areas, in cm^2, of the rectangles $NTMS$, $TLRM$, $PKTN$ and $KQLT$, respectively.

Then the area of the shaded triangle is given by $(w + x + y + z) - (6 + 6 + 5)$, that is, $w + x + y + z - 17$.

The triangle PKN, with area $6\,\text{cm}^2$, is half of the rectangle $PKTN$. Therefore

$$y = 12. \qquad (1)$$

Similarly, the triangles NRS and KQR form halves of the rectangles $NLRS$ and $KQRM$, respectively. Therefore we also have

$$w + x = 12 \qquad (2)$$

and

$$x + z = 10. \qquad (3)$$

It follows from (2) and (3) that

$$w = 12 - x \qquad (4)$$

and

$$z = 10 - x. \qquad (5)$$

We have only three equations for our four unknowns. An extra equation is needed to enable us to deduce the values of w, x, y and z.

So far we have not made full use of the way the four smaller rectangles share some of their sides. Exploiting this enables us to complete the solution.

We have

$$\frac{w}{x} = \frac{MT \times NT}{MT \times TL} = \frac{NT}{TL}$$

and

$$\frac{y}{z} = \frac{TK \times NT}{TK \times TL} = \frac{NT}{TL}.$$

It follows that

$$\frac{w}{x} = \frac{y}{z}. \qquad (6)$$

By using equations (1), (4) and (5) to substitute for y, z and w in equation (6), we obtain

$$\frac{12 - x}{x} = \frac{12}{10 - x}.$$

It follows that

$$(12 - x)(10 - x) = 12x.$$

This last equation may be rearranged to give

$$x^2 - 34x + 120 = 0$$

which is equivalent to

$$(x - 4)(x - 30) = 0.$$

Therefore $x = 4$ or $x = 30$. By equation (4), $x = 30$ implies that $z < 0$ which is not possible because $z\,\text{cm}^2$ is the area of a rectangle.

It follows that $x = 4$. Therefore, by equations (4) and (5), $w = 8$ and $z = 6$. Hence $w + x + y + z - 17 = 8 + 4 + 12 + 6 - 17 = 13$.

We deduce that the area of the shaded triangle is $13\,\text{cm}^2$.

FOR INVESTIGATION

25.1 The diagram on the right shows a rectangle which has been divided into four triangles. The areas of three of the triangles are as shown.

(a) Find a formula for the area, in cm, of the shaded triangle in terms of a, b and c.

(b) Check that your formula has the value 13 when $a = b = 6$ and $c = 5$.